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# Cluster algebras and Markoff numbers

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Abstract We introduce Markoff numbers and reveal their connection to the cluster algebra associated to the once-punctured torus.

## 1 Introduction

Cluster algebras were first introduced and studied in [FZ02] by Fomin and Zelevinsky. The theory of cluster algebras has connections with various areas of mathematics like quiver representations, Lie theory, combinatorics, Teichmüller theory and Poisson geometry.

Markoff numbers, which have been introduced in the work of A. A. Markoff [Mar79, Mar80], are numbers that satisfy the Diophantine Equation

$$a^2 + b^2 + c^2 = 3abc.$$

The study of Markoff numbers is an important subject in combinatorics and plays a role in the theory of rational approximation of irrational numbers.

In this note, we introduce Markoff numbers in the next section. In the third section, we study the cluster algebra arising from the once-punctured torus. The last section is devoted to the connections between Markoff numbers and the cluster algebra.

## 2 Markoff numbers and Markoff triples

**Definition 2.1.** A triple $(a, b, c)$ of positive integers is a **Markoff triple** if $(a, b, c)$ is a solution to the Diophantine Equation

$$a^2 + b^2 + c^2 = 3abc.$$

The numbers $a, b, c$ occurring in a Markoff triple are called **Markoff numbers**.

**Example 2.2.** $(1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 13, 5), (2, 5, 29)$ are Markoff triples, therefore $1, 2, 5, 13, 29$ are Markoff numbers.

**Exercice 2.3.** Is $(13, 2, 5)$ a Markoff triple?

The authors are grateful to their supervisor Professor Thomas Brüstle for his guidance and help.

Each Markoff triple can produce other Markoff triples by means of the following proposition.

**Proposition 2.4.** *Given a Markoff triple* $(a, b, c)$*, set*

$$a' = 3bc - a, \qquad b' = 3ac - b, \qquad c' = 3ab - c.$$

*Then* $(a', b, c), (a, b', c)$ *and* $(a, b, c')$ *are also Markoff triples.*

*Proof.* We only show that $(a', b, c)$ is a Markoff triple. It suffices to verify that

$$a'^2 + b^2 + c^2 = 3a'bc.$$

$$a'^2 + b^2 + c^2 = (3bc - a)^2 + b^2 + c^2 = 9b^2c^2 + a^2 - 6abc + a^2 + b^2$$

$$= 9b^2c^2 + 3abc - 6abc = 9b^2c^2 - 3abc = 3(3bc - a)bc = 3a'bc.$$

Similarly, one can check that $(a, b', c)$ and $(a, b, c')$ are also Markoff triples. □

Therefore one can obtain three new Markoff triples from a given Markoff triple $(a, b, c)$ by **mutations** at each position which we denote as follows:

$$(a, b, c) \xrightarrow{\mu_1} (a', b, c), (a, b, c) \xrightarrow{\mu_2} (a, b', c), (a, b, c) \xrightarrow{\mu_3} (a, b, c').$$

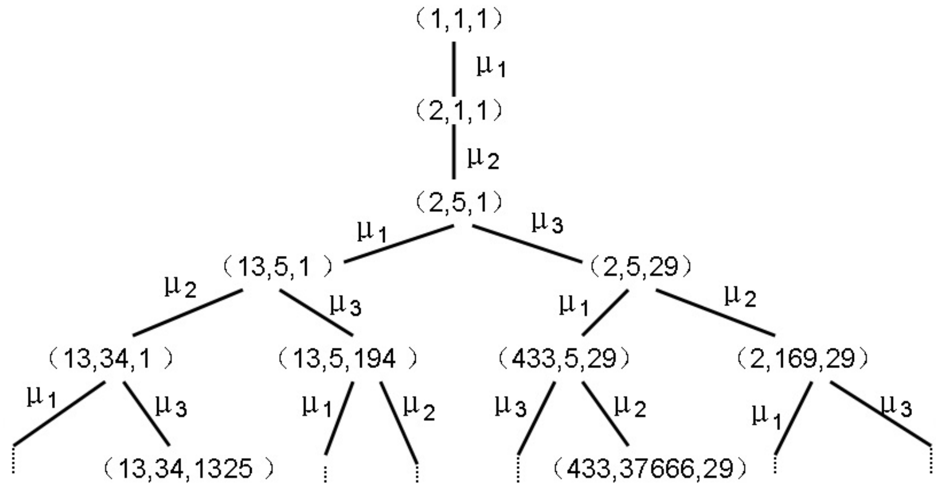
*Remark* 2.5. The mutation is an involution which means $\mu_i^2 =$ Identity, where $i = 1, 2, 3$. For example,

$$\mu_1^2(a, b, c) = \mu_1(3bc - a, b, c) = (3bc - (3bc - a), b, c) = (a, b, c).$$

A. A. Markoff found in [Mar79] that all Markoff triples are connected by mutations:

**Theorem 2.6.** *All Markoff triples are obtained from* $(1, 1, 1)$ *by iterated mutations.*

Note that all permutations of entries of a Markoff triple $(a, b, c)$ are still Markoff triples. For example, $(1, 2, 5), (1, 5, 2), (2, 1, 5), (2, 5, 1), (5, 1, 2), (5, 2, 1)$ are all Markoff triples. In the following, we consider Markoff triples up to permutation and draw a **Markoff tree** with its vertices given by Markoff triples and edges given by mutations:




From the above Markoff tree, one can see that $(433, 37666, 29)$ and $(1, 1, 1)$ are connected as follows:

$$(1,1,1)\overset{\mu_1}{\text{——}}(2,1,1)\overset{\mu_2}{\text{——}}(2,5,1)\overset{\mu_3}{\text{——}}(2,5,29)\overset{\mu_1}{\text{——}}(433,5,29)\overset{\mu_2}{\text{——}}(433,37666,29)$$

which implies

$$\mu_2\mu_1\mu_3\mu_2\mu_1(1,1,1) = (433, 37666, 29).$$

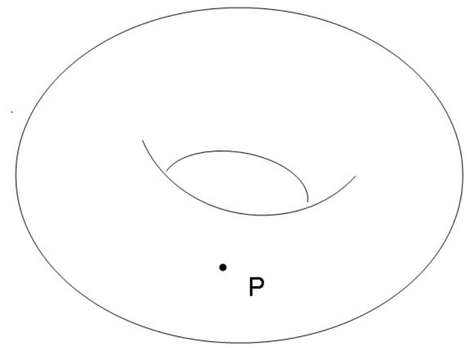
**Exercice 2.7.** Obtain $(14701, 169, 29)$ from $(1, 1, 1)$.

Frobenius [Fro13] claims that every Markoff number appears uniquely as the largest element of a Markoff triple (up to permutation). In fact, this is still a conjecture:

**Conjecture 2.8.** *The Markoff triple is determined by its largest entry (up to permutation).*

## 3 The once-punctured torus

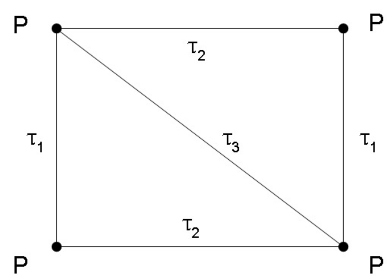
The cluster algebras associated to compact connected oriented Riemman surfaces with marked points have been studied in [SFT08] by Fomin, Shapiro and Thurston. They associated a quiver $Q_\Gamma$ to each triangulation $\Gamma$ of the surface, then the mutations of the quivers correspond to the flips of the triangulations. In this section, we consider the cluster algebra arising from the once punctured surface $T$ as follows :




where $P$ is the only puncture on the surface $T$.

## 3.1 Quiver associated to the once-punctured torus

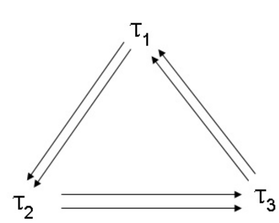
To get the quiver associated to the once punctured torus $T$, we first need to find a triangulation $\Gamma$ of $T$. We cut the torus from $P$ along the vertical line $\tau_1$ and horizontal line $\tau_2$, then we get the following square $S$ :




By adding one more arc $\tau_3$ as in the above picture, one obtains a triangulation $\Gamma = \{\tau_1, \tau_2, \tau_3\}$ of $T$. Recall from [SFT08, LF09, ABCJP10] that a quiver $Q_\Gamma = (Q_0, Q_1)$ is associated to each triangulation $\Gamma$ of the marked surface by the following rule:

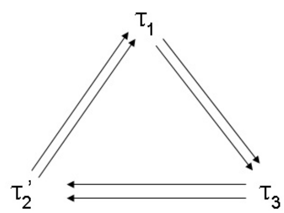
- $Q_0$ is given by the internal arcs of $\Gamma$.
- The set of arrows $Q_1$ is defined as follows: Whenever there is a triangle $\triangle$ in $\Gamma$ containing two internal arcs $a$ and $b$, then there is an arrow $\rho : a \to b$ in $Q_1$ if $a$ is a predecessor of $b$ with respect to clockwise orientation at the joint vertex of $a$ and $b$ in $\triangle$ .

According to the above rule, the quiver $Q_\Gamma$ of the once-punctured torus $T$ can be easily obtained from the square $S$ above as follows:

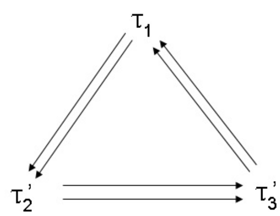


Recall in [BSJ12, FZ02] that the cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, Q)$ with initial seed $\mathbf{x} = \{x_1, x_2, x_3\}, Q = Q_\Gamma$ is a subalgebra of $\mathbb{Q}(x_1, x_2, x_3)$ generated by cluster variables which are obtained by mutations. In order to understand $\mathcal{A}$, we first want to see what happens if we perform mutations at each vertex of $Q$, see [Dup10] for more details about quiver mutations.

- We try to mutate $\tau_2$ first, then the quiver $\mu_2(Q) := \mu_{\tau_2}(Q) = Q^{op}$ is as follows

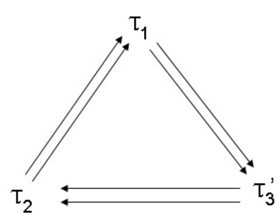



with a new cluster $(x_1, x_2', x_3)$ where $x_2' = \frac{x_1^2+x_3^2}{x_1}$. By mutating at $\tau_3$ again, we get $\mu_3(Q^{op}) := \mu_{\tau_3}(Q^{op}) = \mu_{\tau_3}\mu_{\tau_2}(Q) = Q$ as follows:

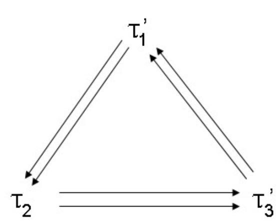



with a new cluster $(x_1, x_2', x_3')$ where $x_3' = \frac{x_1^2+x_2'^2}{x_3}$.

- If we mutate $\tau_3$ first, then $\mu_3(Q) := \mu_{\tau_3}(Q) = Q^{op}$ is as follows:




with a new cluster $(x_1, x_2, x_3')$ where $x_3' = \frac{x_1^2+x_2^2}{x_3}$. Similarly, if we continue mutating at $\tau_1$, then $\mu_1(Q^{op}) := \mu_{\tau_1}(Q^{op}) = \mu_{\tau_1}\mu_{\tau_3}(Q) = Q$ as follows:



with a new cluster $(x_1', x_2, x_3')$ where $x_1' = \frac{x_2^2 + x_3'^2}{x_1}$.

The above procedure implies that the quiver $Q_\Gamma$ is stable under mutations:

**Lemma 3.1.** *$\mu_i(Q) = Q^{op}$ and $\mu_i(Q^{op}) = Q$ where $i = 1, 2, 3$.*

By Fomin-Zelevinsky's work in [FZ02], each cluster variable $f(x_1, x_2, x_3)$ in the cluster algebra $\mathcal{A}$ can be written as

$$f := f(x_1, x_2, x_3) = \frac{g(x_1, x_2, x_3)}{x_1^{d_1} x_2^{d_2} x_3^{d_3}}$$

where $g \in \mathbb{Z}[x_1, x_2, x_3]$ and $d_i \geq 0$ for $i = 1, 2, 3$. Combining the mutation rule of cluster variables with Lemma 3.1, we get the following lemma:

**Lemma 3.2.** *Let $(f_1, f_2, f_3)$ be a cluster of $\mathcal{A}$, then*

$$\begin{aligned}
\mu_1(f_1, f_2, f_3) =& (\frac{f_2^2 + f_3^2}{f_1}, f_2, f_3) \\
\mu_2(f_1, f_2, f_3) =& (f_1, \frac{f_1^2 + f_3^2}{f_2}, f_3) \\
\mu_3(f_1, f_2, f_3) =& (f_1, f_2, \frac{f_1^2 + f_2^2}{f_3}).
\end{aligned}$$

*Proof.* We only prove $\mu_1(f_1, f_2, f_3) = (\frac{f_2^2 + f_3^2}{f_1}, f_2, f_3)$. By Lemma 3.1, the quiver associated to the cluster $(f_1, f_2, f_3)$ must be $Q$ or $Q^{op}$. Without loss of generality, we assume it is $Q$. Therefore, by definition of mutations of clusters

$$f_1' = \frac{1}{f_1}(\prod_{i \to 1 \text{ in } Q} f_i + \prod_{1 \to j \text{ in } Q} f_j) = \frac{1}{f_1}(f_2^2 + f_3^2).$$

This completes the proof. □

# 4 Conclusion

We show in this section the connection between Markoff numbers and the cluster algebra $\mathcal{A}$. The following is our main theorem:

**Theorem 4.1.** *Let $(f_1, f_2, f_3)$ be a cluster in the cluster algebra $\mathcal{A}$. Then $(f_1(1,1,1), f_2(1,1,1), f_3(1,1,1))$ is a Markoff triple.*

*Proof.* We first consider the initial seed $(f_1, f_2, f_3) = (x_1, x_2, x_3)$, then it is obvious that $(f_1(1,1,1), f_2(1,1,1), f_3(1,1,1)) = (1,1,1)$ is a Markoff triple. Therefore the theorem is true for the initial seed.

Assume $(f_1, f_2, f_3)$ is a cluster of $\mathcal{A}$ such that $(f_1(1,1,1), f_2(1,1,1), f_3(1,1,1))$ is a Markoff triple. It suffices to prove that $(f_1'(1,1,1), f_2(1,1,1), f_3(1,1,1))$ is also a Markoff triple where $\mu_1(f_1, f_2, f_3) = (f_1', f_2, f_3)$ since each cluster is obtained by iterated mutations from the initial seed.

Lemma 3.2 implies that $f_1' = \frac{f_2^2+f_3^2}{f_1}$, hence

$$\begin{aligned} f_1'(1,1,1) &= \frac{f_2^2(1,1,1)+f_3^2(1,1,1)}{f_1(1,1,1)} \\ &= \frac{3f_1(1,1,1)f_2(1,1,1)f_3(1,1,1)-f_1^2(1,1,1)}{f_1(1,1,1)} \\ &= 3f_2(1,1,1)f_3(1,1,1)-f_1(1,1,1). \end{aligned}$$

By Proposition 2.4, $(f_1'(1,1,1), f_2(1,1,1), f_3(1,1,1))$ is also a Markoff triple. Similarly, one can prove that $(f_1(1,1,1), f_2'(1,1,1), f_3(1,1,1))$ and $(f_1(1,1,1), f_2(1,1,1), f_3'(1,1,1))$ are also Markoff triples. □

Keeping all the notations above, then we have the following two corollaries:

**Corollary 4.2.** *$f_i(1,1,1)$ is a Markoff number for each $i = 1, 2, 3$.*

**Corollary 4.3.** *There is a bijection between the set of all clusters in $\mathcal{A}$ and the set of all Markoff triples given by*

$$(f_1, f_2, f_3) \longmapsto (f_1(1,1,1), f_2(1,1,1), f_3(1,1,1)).$$

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